Conductance plateaux in the Integer Quantum Hall Effect:

Part I:

1. $\mathcal{H} = \frac{1}{2m}\left(p_x - \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2$.
2. $[\mathcal{H}, p_x] = 0$, so one can diagonalize $\mathcal{H}$ in a basis where p_x is diagonal $\Rightarrow \psi(x, y) = e^{ik_x x}\varphi(y)$.
3. The potential is a quadratic form in y, thus the problem is equivalent to a harmonic oscillator:
$$\frac{1}{2}m\omega_c^2(y - y(k_x))^2$$
where ω_c and $y(k_x)$ need to be determined. One simply needs to match terms in order to determine the constants,
$$\frac{1}{2m}\frac{e^2B^2}{c^2}y^2 = \frac{1}{2}m\omega_c^2 y^2 \quad \Rightarrow \quad \omega_c = \frac{eB}{mc}$$
and
$$2\frac{1}{2m}\hbar k_x\left(\frac{-eB}{c}\right)y = -2\frac{1}{2}m\omega_c^2 y(k_x)y \quad \Rightarrow \quad y(k_x) = \frac{\hbar k_x}{m\omega_c}$$
4. This gives us the energies
$$E(n, k_x) = \hbar\omega_c\left(n + \frac{1}{2}\right).$$
5. Here we compute the expectation value of the current operator in the state $|n, k_x\rangle$:
$$\langle n, k_x|j_x|n, k_x\rangle = -\frac{e}{m}\hbar k_x + \frac{e^2}{mc}By(k_x) = 0.$$
The current $\langle j_y\rangle = 0$ because the ground state corresponds to a harmonic oscillator in the y direction.
6. Here $-\frac{L_y}{2} \leq y(k_x) \leq \frac{L_y}{2}$ leads to
$$-\frac{L_y}{2} \leq \frac{\hbar k_x}{m\omega_c} \leq \frac{L_y}{2}$$
$$-\frac{L_y}{2}\frac{m\omega_c}{\hbar} \leq k_x \leq \frac{L_y}{2}\frac{m\omega_c}{\hbar}$$

Part II:

1. $\mathcal{H} = \frac{1}{2m}\left(p_x - \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2 + eEy$.
2. $[\mathcal{H}, p_x] = 0$, so one can diagonalize $\mathcal{H}$ in a basis where p_x is diagonal $\Rightarrow \psi(x, y) = e^{ik_x x}\varphi(y)$.

3. The potential is a quadratic form in y, thus the problem is equivalent to a harmonic oscillator:

$$\frac{1}{2}m\omega_c^2(y-y(k_x))^2+C$$

where ω_c, $y(k_x)$ and C need to be determined. One simply needs to match terms in order to determine the constants,

$$\frac{1}{2m}\frac{e^2B^2}{c^2}y^2=\frac{1}{2}m\omega_c^2y^2 \quad \Rightarrow \quad \omega_c=\frac{eB}{mc}$$

and

$$\left(2\frac{1}{2m}\hbar k_x\left(\frac{-eB}{c}\right)+eE\right)y=-2\frac{1}{2}m\omega_c^2y(k_x)y \quad \Rightarrow \quad y(k_x)=\frac{\hbar k_x}{m\omega_c}-\frac{eE}{m\omega_c^2}$$

4. To write the energy, one needs to determine the constant shift C:

$$\frac{1}{2}m\omega_c^2y(k_x)^2+C=\frac{\hbar^2k_x^2}{2m}.$$

Using the formula for $y(k_x)$ from above we obtain

$$C=\frac{e\hbar E}{m\omega_c}k_x-\frac{1}{2}\frac{e^2E^2}{m\omega_c^2}.$$

This gives us the energies

$$E(n,k_x)=\hbar\omega_c\left(n+\frac{1}{2}\right)+\frac{e\hbar E}{m\omega_c}k_x-\frac{1}{2}\frac{e^2E^2}{m\omega_c^2}.$$

5. Here we compute the expectation value of the current operator in the state $|n,k_x\rangle$:

$$\langle n,k_x|j_x|n,k_x\rangle=-\frac{e}{m}\hbar k_x+\frac{e^2}{mc}By(k_x)=-ec\frac{E}{B}.$$

The current $\langle j_y\rangle=0$ because the ground state corresponds to a harmonic oscillator in the y direction.

6. Here $-\frac{L_y}{2}\leq y(k_x)\leq\frac{L_y}{2}$ leads to

$$-\frac{L_y}{2}\leq\frac{\hbar k_x}{m\omega_c}-\frac{eE}{m\omega_c^2}\leq\frac{L_y}{2}$$

$$\left(-\frac{L_y}{2}+\frac{eE}{m\omega_c^2}\right)\frac{m\omega_c}{\hbar}\leq k_x\leq\left(\frac{L_y}{2}+\frac{eE}{m\omega_c^2}\right)\frac{m\omega_c}{\hbar}$$

Part III:

1. $\mathcal{H}=\frac{1}{2m}\left(p_x-\frac{eB}{c}y\right)^2+\frac{1}{2m}p_y^2+eEy+\frac{1}{2}m\omega_0^2y^2.$
2. $[\mathcal{H},p_x]=0\Rightarrow\psi(x,y)=e^{ik_xx}\varphi(y)$. The potential is still a quadratic form in y, thus the problem is still equivalent to a harmonic oscillator:

$$\frac{1}{2}m\omega^2(y-y(k_x))^2+E_t$$

where ω and $y(k_x)$ need to be determined.

Solving one finds,

$$\omega^2 = \omega_c^2 + \omega_0^2 \quad \text{or} \quad \omega = \sqrt{\omega_c^2 + \omega_0^2}$$

and

$$m\omega^2 y(k_x) = \frac{1}{2m}\hbar k_x + \frac{2eB}{c} - eE$$

leading to

$$\Rightarrow y(k_x) = \hbar k_x \frac{\omega_c}{m\omega^2} - \frac{eE}{m\omega^2}$$

or

$$y(k_x) = \frac{\hbar k_x}{m}\frac{\omega_c}{\omega_c^2 + \omega_0^2} - \frac{eE}{m(\omega_c^2 + \omega_0^2)}.$$

3. For the energy we still need the constant:

$$\frac{1}{2}m\omega^2 y(k_x)^2 + E_t = \frac{\hbar^2 k_x^2}{2m}$$

$$\begin{aligned}
\Rightarrow E_t &= \frac{\hbar^2 k_x^2}{2m} - \frac{1}{2}m\omega^2 y(k_x)^2 \\
&= \frac{\hbar^2 k_x^2}{2m} - \frac{1}{2}m\omega^2 \left(\hbar^2 k_x^2 \frac{\omega_c^2}{m^2\omega^4} - \frac{2\hbar k_x eE\omega_c}{m^2\omega^4} + \frac{e^2E^2}{m^2\omega^4}\right) \\
&= \frac{\hbar^2 k_x^2}{2m} - \frac{1}{2m}\hbar^2 k_x^2 \frac{\omega_c^2}{\omega^2} + \hbar k_x \frac{eE}{m}\frac{\omega_c}{\omega^2} - \frac{e^2E^2}{2m\omega^2}
\end{aligned}$$

or

$$E_t = \frac{\hbar^2 k_x^2}{2m}\frac{\omega_0^2}{\omega^2} + \hbar k_x \frac{eE}{m}\frac{\omega_c}{\omega^2} - \frac{e^2E^2}{2m\omega^2}.$$

Thus,

$$E(n, k_x) = \hbar\omega\left(n + \frac{1}{2}\right) + E_t$$

$$E(n, k_x) = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_x^2}{2m}\frac{\omega_0^2}{\omega^2} + \frac{eE\hbar\omega_c k_x}{m\omega^2} - \frac{e^2E^2}{2m\omega^2}.$$

We have,

$$\frac{\partial E}{\partial k_x} = \frac{\partial E_t(k_x)}{\partial k_x}$$

Moreover, E_t is a quadratic function of k_x with a positive quadratic term, thus E has a minimum in k_x:

$$\frac{\hbar^2\omega_0^2}{m\omega^2}k_x + \frac{eE\hbar\omega_c}{m\omega^2} = 0$$

$$\Rightarrow k_{x,\min} = -\frac{eE}{\hbar}\frac{\omega_c}{\omega_0^2}. \tag{1}$$

4. $-\frac{L_y}{2} \leqslant y(k_x) \leqslant \frac{L_y}{2}$ leads to,

$$\left(-\frac{L_y}{2} + \frac{eE}{m\omega^2}\right)\frac{m\omega^2}{\hbar\omega_c} \leqslant k_x \leqslant \left(\frac{L_y}{2} + \frac{eE}{m\omega^2}\right)\frac{m\omega^2}{\hbar\omega_c}. \tag{2}$$

5. Inserting k_x given in Eq. (1) in the inequality (2) we obtain,

$$|E| \leqslant \frac{L_y m\omega_0^2}{2e}.$$

6. The current is given by,

$$< j_x >= -\frac{e\hbar k_x}{m} + \frac{e^2 B}{mc} y(k_x) \tag{3}$$

leading to,

$$< j_x >= -\frac{e\hbar k_x}{m}\frac{\omega_0^2}{\omega^2} - \frac{e^2 \omega_c E}{m\omega^2}. \tag{4}$$

Inserting the expression of $k_{x,\min}$ in Eq. (1) one gets,

$$< j_x >= 0.$$

7. It is useful to draw the dispersion. When we start filling a band, the current is 0 and remains 0 until we have reached the left boundary for k_x, namely $k_x^{\min}$. Remember that we fill the band symmetrically around $k_{x,\min}$. Thus the conductance remains flat for a while. Then we start filling the states on the right of $2k_{x,\min} - k_x^{\min}$ which leads to a net current.